# Algorithm for Energy-Derived Potential Flow Hydrodynamic Coefficients

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**An algorithm is described to compute steady and unsteady hydrodynamic coefficients from added mass tensors without the need to develop highly intense and system dependent algebraic expressions and computer codes, especially when complex motions are admitted. Lagrange's equation is applied in fixed coordinates. Coordinate transformations are accomplished to obtain an efficient algorithm for six-degrees-of-freedom coefficients for bodies hydrodynamically defined with time varying added mass tensors. In addition to infinite fluid, the algorithm is applicable to restricted classes of emerging body and free surface conditions.**

## Nomenclature

$A_{i,j}$ = hydrodynamic added mass tensor
$e_p$ = coordinate in body frame, $p=1, 3$
$F_k$ = hydrodynamic force and moment, body coordinates, $k=1, 6$
$f(k,k)$ = 0 for all $k$
$f(k,m)$ = $6-R$ for $k \neq m$, where $R$ is the remainder of $(k+m)/3$, or the principal residue of $k+m$ (mod 3)
$I_{i,j}$ = 1 if $i=j$, 0 otherwise, $i=0, 6$; $j=1, 6$
$n$ = normal on infinite plane
$n_j$ = direction cosine between boundary motion, with velocity $u_j$ and surface normal
$q_k$ = generalized coordinate defining body position in the moving frame, $k=1, 6$
$S_{m,k}$ = $6\times 6$ coefficient matrix (defined in text), $m$ and $k$ are row and column indexes, respectively
$S$ = body surface
$T$ = total kinetic energy in fluid, relative to inertial frame
$t$ = time
$u_i$ = body scalar velocity component, body coordinate system, relative to inertial frame, $i=1, 6$
$X_n$ = coordinates in moving frame system, $X_3$ is normal to frame, $n=1, 3$
$Z_\ell$ = arbitrary coordinates defining body position relative to moving frame, $\ell=1, 6$
$\Phi_i$ = unit velocity potential associated with scalar velocity $u_i$ having unit magnitude, $i=1, 6$
$\rho$ = fluid density
$\dfrac{\partial Z_\ell}{\partial X_n}$ = moving to relative coordinate $6\times 6$ transformation matrix, $\ell$ and $n$ are column and row designators, respectively
$\dfrac{\partial X_n}{\partial e_k}$ = body to moving coordinate $6\times 6$ transformation matrix, $k$ and $n$ are column and row designators, respectively
$(\dot{\ })$ = time rate of change of variable in coordinate system associated with variable

## Introduction

THE present paper is concerned with the innovative and practical application of established classical hydrodynamic theoretical relations, to the task of computing generalized hydrodynamic force and moment coefficients from sets of potential flow solutions for a body moving through, or in close proximity to a moving, infinite plane. Included are those conditions where the Lagrange equation may be used to describe force and moment on the body using added mass tensors computed from integration of the velocity potential functions over the body surface. This admits inviscid fluid dynamic modeling of bodies emerging through apertures in infinite, moving planes and two restricted classes of free-surface modeling where limiting conditions of Froude number are specified.

The classical texts typically describe the general relations associated with the use of the Lagrange equation and the added mass tensor in terms of descriptions that are limited to presenting the equations of motion, in an infinite fluid, in terms of energy. Kelvin-Kirchoff equations are elegant and compact, but they do not address the applications complexity introduced when the added mass tensor is substituted into the energy term under conditions of body rotation. The Kelvin-Kirchoff equations were extended by Chow, Hou, and Landweber[1] to conditions for a body emerging from a moving, infinite plane where a particular selection of limited coordinates is used to describe the body position. Expressions for force and moment coefficients were shown for planar motion conditions where three velocity and two position coordinates were used to describe the dynamical system. The algebraic expansions, software coding, and verification are readily manageable when using a limited number of kinematic variables. However, the implementation tasks are formidable for completely general systems when as many as twelve coordinates are used to describe the velocities and the position of the body. Since the terms in the force and moment equations are dependent on the particular choice of coordinates used to describe position, there is no general set of verified force and moment equations expressed in algebraic form.

The extension that this paper provides to the work of Chow, et al.,[1] is the introduction of a complete set of arbitrary coordinates to describe the position of the body relative to a fixed position on the moving infinite plane, and the development of a summation algorithm that defines the contributors to the hydrodynamic coefficients in terms of those quantities readily available from sets of potential flow solutions. The need is thus avoided to explicitly derive the extended Kelvin-Kirchoff equations for a particular selection of relative position coordinates, with the resulting complex algebraic expansions.

In the general case, there can be as many as 162 nontrivial coefficients, and each coefficient can have many independent contributions to it. As hydrodynamic modeling becomes more sophisticated, dynamic simulation of non-simplistic body geometries emerging from moving underwater vehicles,

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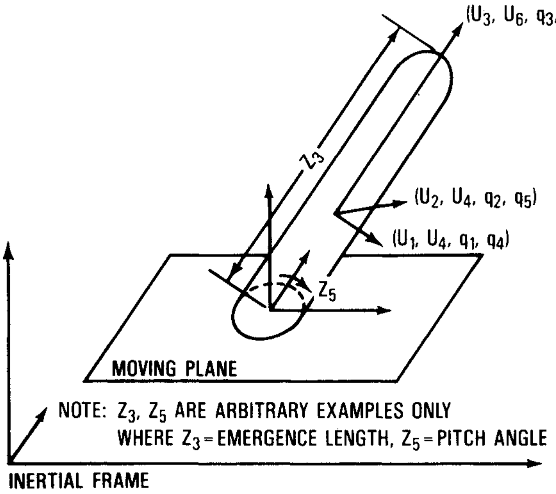


**Fig. 1 Coordinate systems.**

poses increasingly complex demands on the number of degrees-of-freedom required. The practical goal here is to admit all six-degrees-of-freedom of body motion and to allow an arbitrary specification of coordinates to define body position relative to a reference frame fixed to the moving infinite plane.

Energy and pressure comprise the two methods available to compute hydrodynamic loads. Energy derived force and moment coefficients are preferred over the more difficult to obtain pressure derived coefficients. When it is necessary to use pressure integration to determine distributed forces and moments, it is still advisable to perform a redundant energy computation for the total hydrodynamic force and moment coefficients to verify the pressure integration results. When only total forces and moments are required, the energy method is much less demanding in the numerical precision required of the potential flow solutions. The energy method requires finite differencing only at the added mass level while the pressure option requires detailed differencing of velocity potentials at the surface mesh level for both spatial gradients on the body and variations on the body with respect to changes in relative position. Since the mesh coordinate system typically is fixed to the body, the differencing of potentials at mesh points just penetrating the infinite plane requires considerable care.[2]

The method presented herein derives from the application of summation algorithms and indexing conventions synthesized to combine two governing relations used to express hydrodynamic force and moment coefficients in terms of those quantities conveniently available from potential flow solutions. These two equations are:

1) Lagrange—to relate force and moment on the fluid from the body to time and spacial variations in fluid kinetic energy.

2) Hydrodynamic added mass tensor—to relate the total time-dependent kinetic energy to the coordinates describing body position and motion.

While not used in the algorithm explicitly, the Kirchoff equation is applied to relate the unit velocity potentials to the added mass tensor. Arbitrary, but sufficient coordinates are specified to fix the position of the body on the infinite plane. The resulting algorithm is compact and avoids the need to explicitly derive algebraic expressions for the extended Kelvin-Kirchoff equations and the resulting complex expressions when the added mass tensor substitutions are made. Software verification is greatly simplified and the algorithm is applicable for arbitrary selections of body position coordinates. The particular choice of coordinates used to describe the body position is specified with two $6 \times 6$ orthogonal transformation matrices used by the algorithm.

This algorithm is restricted to conditions where either the velocity potential, or the normal gradient of the velocity potential is equal to zero at all points on the infinite plane. The added mass tensor is developed from integration of the velocity potential over the body surface where it is assumed that there is no contribution to the added mass tensor from the infinite plane. When modeling the emergence of a body from a moving rigid plane, an imposed potential flow boundary condition is that the velocity potential normal gradient is zero. In free surface image plane modeling, the assumption of a very low Froude number requires the velocity potential normal gradient on the infinite plane to be zero. Conversely, the assumption of a very high Froude number requires the velocity potential to be zero on the plane. The following sections describe theoretical background, algorithm formulation and details required to correctly apply the index and transformation definitions. The last section discusses algorithm verification.

## Theoretical Background

The objective is to formulate a system of velocity and position coordinates that are arbitrary yet allow compliance with the requirements of the Lagrange equation and kinetic energy description using an added mass tensor. The Lagrange equation requires definition of position with independent generalized coordinates that are referenced to a coordinate system moving with constant linear velocity. The description of energy with the added mass tensor requires that the scalar velocities be referenced to a coordinate system where the undisturbed fluid velocity is zero. There is no requirement that these two reference coordinate systems be coincident or moving with the same velocity, other than in a direction normal to the infinite plane. Also there is no requirement that we define position with independent generalized coordinates, only that we use the latter when applying the Lagrange equation. Figure 1 shows the body, the moving plane reference frame, the inertial system, and the coordinates describing body velocity and position.

The velocity of a fluid particle not disturbed by the body motion is zero, according to the definition of $u_i$. $Z_\ell$ is an arbitrary set of coordinates selected to conveniently define body position relative to the moving frame coordinate system. Typically, this can be three translations and three Euler angles. Alternatively, emergence length and angle are convenient selections for planar motion.

Coordinate selections generally exist that can render certain hydrodynamic and/or relative position coordinates trivial. However, retention of generality promotes identification of recursive relations for algorithm synthesis and provides unrestricted coordinate selection. A set of generalized position coordinates, $q_k$, is instantaneously aligned with the body coordinate system and is used to represent position relative to the translating frame. The instantaneous coordinate system associated with the generalized coordinates does not rotate.

The motions of the plane are restricted to constant velocity in the transverse directions. Infinite plane transverse velocity has no significance. The $q_k$ array represents a legitimate set of generalized coordinates for the Lagrange equation by representing independent quantities defining position relative to a reference frame moving with constant velocity. The $u_i$ array represents a legitimate set of generalized coordinates to describe the fluid kinetic energy by referencing a coordinate system where undisturbed fluid velocity is zero. The force acting on the fluid system is expressed with the Lagrange equation

$$F_k = \frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \tag{1}$$

Equation (1) represents all sources of force on the fluid dynamic system that affect the fluid kinetic energy. These sources include force from both the body and the infinite plane on the fluid. Since the purpose of this paper is to describe the reaction force on the body, it is necessary that Eq. (1) represent only dynamic loads from the body. This condition is satisfied when either one of the following boundary conditions is enforced at the infinite plane for each of the potential flow solutions:

$$\Phi_i = 0 \qquad (i=1,\ 6) \tag{2}$$

$$\frac{\partial \Phi_i}{\partial_n} = 0 \qquad (i=1,\ 6) \tag{3}$$

## Algorithm Formulation

The objective is to synthesize computer-oriented expressions that allow added mass tensor transformation directly to hydrodynamic coefficients, without developing the extended Kelvin-Kirchoff relations, and without performing the massive algebraic manipulations associated with a general application of the Lagrange equation using moving coordinates.

Lets begin by reviewing what is generally available from the Neumann problem. Potential flow solutions are computed for each generalized velocity for a discrete set of positions for each of the relative position coordinates. From this set, the hydrodynamic added mass tensor is computed for each unique position using the Kirchoff equation

$$A_{i,j} = -\rho \int_S \Phi_i n_j \mathrm{d}S \tag{4}$$

where the $i,j$ indices refer to the generalized coordinates used to describe velocity. A unique $A_{i,j}$ added mass tensor exists for each unique set of $Z_\ell$, describing the relative positions. The total kinetic energy in the fluid is related to the body velocities with the added mass tensor and the body velocity scalars

$$2T = \sum_{i=1}^{6} \sum_{j=1}^{6} A_{i,j} u_i u_j \tag{5}$$

These scalars are referenced to the moving body coordinate system to facilitate computation and use of the hydrodynamic coefficients in dynamic analysis. Consequently, $\dot{u}$ does not represent inertial acceleration.

All the governing equations, effects of rotating coordinate systems on scalar acceleration, and coordinate system transformations are expressible as summations over added mass, and transformation matrices identifiable from the particular choice of relative position coordinates used.

The use of generalized expressions for all the coordinates allows identification of recursive relations and patterns which may be exploited to synthesize the following algorithm:

$$\begin{aligned} 2F_k = \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{m=1}^{6} \Big[ & A_{i,j} S_{m,k} \{ I_{m,k} (\dot{u}_i I_{j,m} + \dot{u}_j I_{i,m}) \\ & + u_{f(m,k)} (u_i I_{j,m} + u_j I_{i,m}) \} - A_{i,j} (u_i u_k S_{j,m} I_{f(j,m),k} \\ & + u_j u_k S_{i,m} I_{f(i,m),k}) + \sum_{\ell=1}^{6} \sum_{n=1}^{6} \frac{\partial A_{i,j}}{\partial Z_\ell} \frac{\partial Z_\ell}{\partial X_n} \Big\{ -u_i u_j \frac{1}{6} \frac{\partial X_m}{\partial e_k} \\ & + \sum_{p=1}^{6} \frac{\partial X_n}{\partial e_p} u_p (u_i I_{j,m} + u_j I_{i,m}) \Big\} \Big] \end{aligned} \tag{6}$$

where

$$S_{m,k} = \begin{vmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{vmatrix}$$

The detailed derivation is available from the author on request. The algorithm compactly expresses total hydrodynamic force and moment in terms of quantities readily identified from a) added mass tensors, b) added mass tensor spacial variations, with respect to $Z_\ell$, c) index algorithms, and d) coefficient matrices.

The algorithm expresses force and moment directly in terms of the scalar velocities and accelerations, $u_i$ and $\dot{u}_i$ and the added mass tensors. Consequently, the coefficients associated with each $u_i u_j$ pair and $\dot{u}_i$ are easily identified. The structure of the algorithm allows the use of nested DO loops to orderly load hydrodynamic coefficient arrays using minimum software. In addition, the algorithm can be used to methodically construct algebraic expressions for the hydrodynamic coefficients. The partial derivitives which transform moving to relative and body to moving coordinates are $\partial Z_\ell / \partial X_n$ and $\partial X_n / \partial e_k$, respectively. Note carefully the row and column assigned to each. An example is provided for those transformations in the next section.

## Algorithm Verification

The six-degree-of-freedom energy algorithm has been verified by comparing its results with coefficients computed by Chow, et al.,[1] for planar motion conditions. The corresponding transformation matrices contain 2 and 5 nontrivial terms for this selection of relative position coordinates (emergence length and angle). These are

$$\frac{\partial Z}{\partial X}(3,\ 3) = \frac{1.0}{\cos\theta_2} \tag{7a}$$

$$\frac{\partial Z}{\partial X}(5,\ 5) = 1.0 \tag{7b}$$

and

$$\frac{\partial X}{\partial e}(1,\ 1) = \frac{\partial X}{\partial e}(3.3) = \cos\theta_2 \tag{8a,b}$$

$$\frac{\partial X}{\partial e}(1,\ 3) = \sin\theta_2 \tag{8c}$$

$$\frac{\partial X}{\partial e}(3,\ 1) = -\sin\theta_2 \tag{8d}$$

$$\frac{\partial X}{\partial e}(5,\ 5) = 1.0 \tag{8e}$$

The hydrodynamic coefficients computed with the algorithm agree with those computed with the terms in Eqs. (32-34) of Ref. 1.

## Summary

A generalized method has been developed to compactly express hydrodynamic force and moment coefficients for a body moving with six-degrees-of-freedom under the influence of a moving infinite plane. The computationally oriented algorithm allows arbitrary and convenient definition of the body position relative to the infinite plane. The need to explicitly derive coordinate system-dependent extended Kelvin-Kirchoff equations is avoided as is the need to perform massive transformations and back substitutions of the added mass tensor. Only two $6 \times 6$ transformations matrices need to be redefined when coordinate system changes are made.

## References

[1]Chow, S. K., Hou, A. Y., and Landweber, L., "Hydrodynamic Forces and Moments Acting on a Body Emerging from an Infinite Plane," *Physics of Fluids*, Vol. 19, Oct. 1976, pp. 1439-1449.

[2]Burgdorf, O., "Multiple Body Six Degrees of Freedom Potential Flow Pressure Equation," *Journal of Hydronautics*, Vol. 14, Jan. 1980, pp. 19-24.